

Nonuniversal behavior for aperiodic interactions within a mean-field approximation

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We study the spin-1/2 Ising model on a Bethe lattice in the mean-field limit, with the interaction constants following one of two deterministic aperiodic sequences, the Fibonacci or period-doubling one. New algorithms of sequence generation were implemented, which were fundamental in obtaining long sequences and, therefore, precise results. We calculate the exact critical temperature for both sequences, as well as the critical exponents β , γ , and δ . For the Fibonacci sequence, the exponents are classical, while for the period-doubling one they depend on the ratio between the two exchange constants. The usual relations between critical exponents are satisfied, within error bars, for the period-doubling sequence. Therefore, we show that mean-field-like procedures may lead to nonclassical critical exponents.

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Considerable attention has been devoted to the investigation of systems displaying inhomogeneous or disordered interactions [1]. These systems have experimental relevance, since many of the materials found in nature come with impurities. Moreover, modern techniques are able to build materials with controlled composition, such that two or more different atoms are combined in a given order [2]. Theoretically, one possible question is to what extent the introduction of quenched disorder may affect the critical behavior of an originally pure system [1]. The answer to this question for random disorder is given by the Harris criterion [3]: if the pure system's specific-heat exponent α is positive (negative), the critical behavior of the disordered model is different from (the same as) that of the pure model.

The discovery of quasicrystals [2] has motivated the introduction of another kind of inhomogeneity in a system: its interactions are now modulated by aperiodic sequences, obtained from, for example, deterministic substitution rules. These sequences have been a subject of intense research in recent years [4]; numerical [5] as well as analytical results [6] have been obtained and the so-called Harris-Luck criterion is available [7]. According to this criterion, the relevance of a given aperiodic sequence is connected to the crossover exponent $\phi = 1 - d_a \nu_0 (1 - \omega)$, where d_a is the dimension in which the aperiodic sequence acts, ν_0 is the correlation length's critical exponent in the pure model, and ω is defined through $g \sim N^\omega$. Here, g is the fluctuation in numbers of a given letter of the sequence, and N is the length of the sequence (see below). Generally speaking, the greater the fluctuation of the sequence the easier it is to move the critical behavior of the system away from the homogeneous case. We can use many approaches to address the influence of aperiodicity on critical behavior. The mean field is perhaps the most used approximation in condensed matter problems, ranging from complex fluids [8] and superconductivity [9] to Bose-Einstein condensation [10] and magnetism [11,12]. One possible realization of a mean-field picture is the Bethe lattice: the critical exponents for homogeneous models in

this geometry are classical (as for the usual mean-field approximations), namely, $\beta = 1/2$, $\gamma = 1$, and $\delta = 3$, for example. In the present case, we arrive at the surprising result that aperiodic modulation of exchange constants may change the critical exponent of a magnetic system *within a mean-field approximation*. Therefore, the application of this approach seems to be generally useful in the study of critical phenomena in complex systems.

Our goal in this paper is twofold: to investigate whether there are changes in the universality class of magnetic systems with aperiodic interactions in high-dimensional systems and to determine the existence or not of nonclassical critical exponents within a mean-field framework. To achieve this, we define the Ising model on a Bethe lattice, which has proven to be a good approximation to the critical behavior in dimension 3 or above. In this geometry, no closed loops are allowed; see Fig. 1 for an example of a portion of a Bethe lattice with $z=3$. The model we study is defined by the Hamiltonians:

$$\mathcal{H}_{1/2} = - \sum_{\langle i,j \rangle} J_n s_i s_j, \quad s_i = \pm 1, \quad (1)$$

where the sum is over nearest-neighbor spins and J_n is the exchange constant between sites in generations n and $n+1$ and may have different values, depending on the generation and the corresponding letter in the aperiodic sequence (although all interactions between the same pair of generations have the same value).

The exchange constant J_n is chosen according to the corresponding letter in the aperiodic sequence. The construction of the aperiodic sequence is made in the Bethe lattice from the exterior to the interior, i.e., the first letters of the sequence correspond to the interactions between generations in the exterior of the lattice. The sequences we have studied are the following.

(i) The Fibonacci sequence, built from the substitution rules $A \rightarrow AB$, $B \rightarrow A$. The first stages of this sequence are $A \rightarrow AB \rightarrow ABA \rightarrow ABAAB \rightarrow ABAABABA$. This last finite sequence corresponds to the following sequence of interaction constants: $J_A J_B J_A J_A J_B J_A J_B J_A$.

(ii) The period-doubling sequence: in this case the substi-

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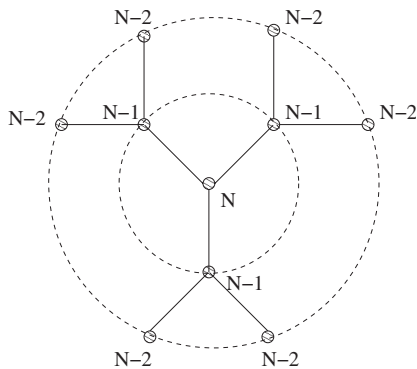


FIG. 1. Portion of a Bethe lattice with coordination number $z = 3$. Spins are represented by black dots. Generations are linked by dashed lines (which do not represent interactions between spins) and the interaction constants between different generations (continuous lines) may have two different values, J_A or J_B .

tution rules are $A \rightarrow AB$ and $B \rightarrow AA$. The first letters of the interaction constant sequence are $J_A J_B J_A J_A J_B J_A J_B$.

The geometrical characteristics of aperiodic sequences can be obtained from the substitution matrix \mathcal{M} , which connects the number of letters A and B after one application of the iteration rule, namely,

$$\begin{pmatrix} N_A^{(n+1)} \\ N_B^{(n+1)} \end{pmatrix} = \mathcal{M} \begin{pmatrix} N_A^{(n)} \\ N_B^{(n)} \end{pmatrix}, \quad (2)$$

where $N_\zeta^{(n)}$ is the number of letters ζ (A or B) after n iterations of the substitution rule. In particular, the total number of letters, $N^{(n)}$, grows exponentially with the number of iterations n :

$$N^{(n)} \sim \lambda_1^n \quad \text{as } n \rightarrow \infty, \quad (3)$$

where λ_1 is the greater eigenvalue of \mathcal{M} . The fraction of letters A (B), p_A (p_B), in the infinite word, i.e., after n applications of the iteration rules, for large n , is proportional to the first (second) entry of the eigenvector corresponding to the greater eigenvalue of \mathcal{M} (see Ref. [13] for more details). The fluctuation $g^{(n)}$ of a given letter, say A , is defined as

$$g^{(n)} = N_A^{(n)} - p_A N^{(n)}. \quad (4)$$

It is possible to show that $g \sim N^\omega$, where $\omega = \log_{10} |\lambda_2| / \log_{10} \lambda_1$ (λ_2 is the smaller eigenvalue of the substitution matrix). In a linear chain, the fluctuation is greater for the period-doubling sequence than for the Fibonacci one (see below for a discussion of this point).

The solution on a Bethe lattice is obtained from a mapping between partial magnetizations of two consecutive generations; see [14] for a detailed calculation of this mapping. In the mean-field limit, $z \rightarrow \infty$, $J_n \rightarrow 0$, such that the product $\tilde{J}_n \equiv z J_n$ is finite, and for external magnetic field h , the expression simplifies a great deal and we obtain

$$m_{n+1} = \tanh(\tilde{K}_n m_n + H), \quad (5)$$

where m_i is the partial magnetization in generation i , $\tilde{K}_n \equiv \tilde{J}_n / k_B T$, T is the temperature, k_B is the Boltzmann constant,

and $H = h / k_B T$. Note that we took the mean-field limit in order to simplify the equations and calculate the critical temperature analytically.

The critical points of the model studied here are at zero external field. The exact critical temperature is given by the stability limit of the paramagnetic phase, obtained from

$$\left. \frac{dm_n}{dm_1} \right|_{\{m\}=0} = 1, \quad (6)$$

where m_n is the magnetization of spins in generation n ($n \rightarrow \infty$), m_1 is the magnetization of spins in generation 1, and $\{m\}=0$ means that the derivatives are taken at the point where the magnetization of all generations from 1 to $n-1$ is zero. We then obtain

$$\frac{k_B T_c}{\tilde{J}_A} = (1+r)^{p_B}, \quad (7)$$

where T_c is the critical temperature, $(1+r) \equiv \tilde{J}_B / \tilde{J}_A$, and p_B is the fraction of interactions J_B in the thermodynamic limit. As discussed above, this fraction is proportional to the second entry of the eigenvector corresponding to the largest eigenvalue of the matrix \mathcal{M} [13]. It is important to know T_c exactly (or with a very good precision) in order to obtain reliable values for the exponent β .

The magnetization is not uniform; rather, it follows an aperiodic sequence. Therefore, we will calculate and discuss the behavior of the mean magnetization, defined by

$$\bar{m} \equiv \frac{1}{N} \sum_{i=1}^N m_i, \quad (8)$$

where m_i is the partial magnetization in generation i , calculated through Eq. (5). This is also the physical quantity that would be accessible to experiments. In order to calculate γ , we define the zero-field susceptibility $\chi = \partial \bar{m} / \partial h$, related to the mean magnetization.

At this point, it is worth recalling that, for systems such that the length-scaling parameter, in the renormalization-group context, *cannot* assume any value, thermodynamic quantities may follow log-periodic scaling laws [15–17]. The mean magnetization at zero external magnetic field, for example, is written as

$$\bar{m} = t^\beta P[\log_{10}(t)], \quad (9)$$

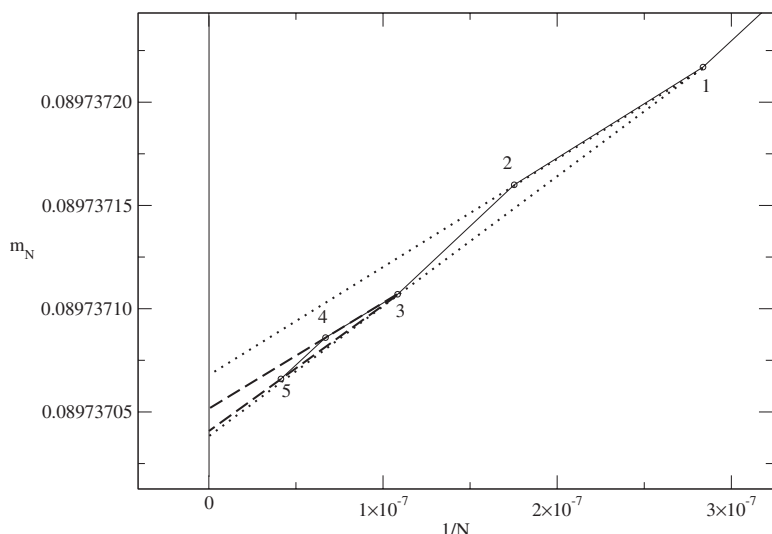
where $t = (T_c - T) / T_c$ is the reduced temperature and $P[x]$ is a periodic function of x . Therefore, the logarithmic derivative of \bar{m} at $h=0$ is given by

$$\frac{d \log_{10} \bar{m}}{d \log_{10} t} = \beta + \tilde{P}[\log_{10}(t)], \quad (10)$$

where $\tilde{P}[x]$ is also a periodic function of x . In an analogous way,

$$\frac{d \log_{10} \bar{m}}{d \log_{10} h} = 1 / \delta + \tilde{Q}[\log_{10}(h)], \quad T = T_c, \quad (11)$$

and



$$\frac{d \log_{10} \chi}{d \log_{10} t} = \gamma + \tilde{R}[\log_{10}(t)], \quad h = 0, \quad (12)$$

where \tilde{Q} and \tilde{R} are periodic functions of their arguments. Our strategy is to calculate the derivatives, do a best fit to a log-periodic function plus a constant, and obtain the values of the respective exponents from this fit. Some points are worth stressing. In order to calculate this derivative numerically, we need two values of \bar{m} in the thermodynamic limit. These are obtained in the following way. We start with an arbitrary value of m_1 and iterate Eq. (5) many times, keeping track of the mean magnetization throughout the process (the number of iterations we used to obtain these results are about 10^8 for the Fibonacci sequence and $\sim 10^{11}$ for the period-doubling sequence). After a transient and a sufficient number of iterations, the mean magnetization fluctuates around a monotonic trend (see Fig. 2), which does not depend on the initial value m_1 . From these values, we can extrapolate to the thermodynamic limit. This procedure can be done for zero or nonzero magnetic field; therefore, we can calculate \bar{m} as a function of t or h , and the susceptibility χ .

In Fig. 2 we depict m_N/N for the Fibonacci sequence for $h=0$, where N is the total number of generations of the Bethe lattice or, equivalently, the number of letters generated for the sequence, together with the extrapolation procedure. We take two consecutive pairs of values of m_N and make a linear extrapolation of each pair to the limit $1/N \rightarrow \infty$. The extrapolated value is taken as the mean of the two values obtained and its error is estimated as half the interval. This is an overestimation of the error but we are sure the true extrapolated value is within the interval. We have compared our procedure with more formal extrapolations like the so-called VBS [18] and BST [19]. The extrapolated value is the same, although the error estimation is somewhat arbitrary for the VBS algorithm and clearly underestimated for the BST one. With this procedure, we were able to calculate the derivative in Eqs. (10)–(12) in the thermodynamic limit.

A crucial point should be mentioned here: only using new algorithms were we able to generate very long sequences for the period-doubling case. With these algorithms, we need to

store only N letters, in order to use an N^2 -letter sequence. Long sequences allow for a reliable and precise extrapolation to the thermodynamic limit (see below). The algorithm for the period-doubling sequence works as follows. If an N^2 -letter sequence, with $N=2^n$, n integer, is generated and it is written in lines of length N , one over the other, it is easy to see that the first $N-1$ columns have the same letter as in the first line of the corresponding column, while the last column repeats the first line if n is even or repeats the first line with A and B exchanged if n is odd. Therefore, one needs to generate and store only the first line, with N letters, to be able to work with an N^2 sequence. This algorithm, although with more complicated rules, can be used for other sequences, which allow for economy in time and space.

Our results can be summarized in the following figures, where we depict the derivatives in Eqs. (10)–(12). These equations tell us that the logarithmic derivatives are represented by periodic functions of $\log_{10} t$ or $\log_{10} h$, with a mean given by the value of the respective exponent. For the

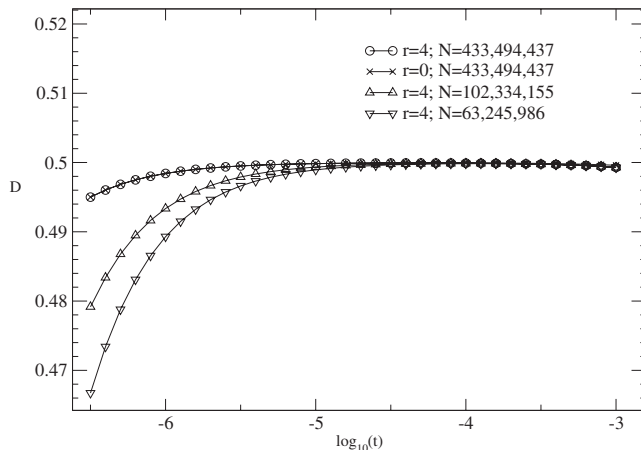


FIG. 3. Logarithmic derivative of the magnetization as a function of the reduced temperature for the Fibonacci sequence. We show the behavior for different values of r and for different numbers of generations of the Bethe lattice. For comparison, the result for the uniform model ($r=0$) is also depicted. It is clear that β is the same for the uniform and aperiodic models.

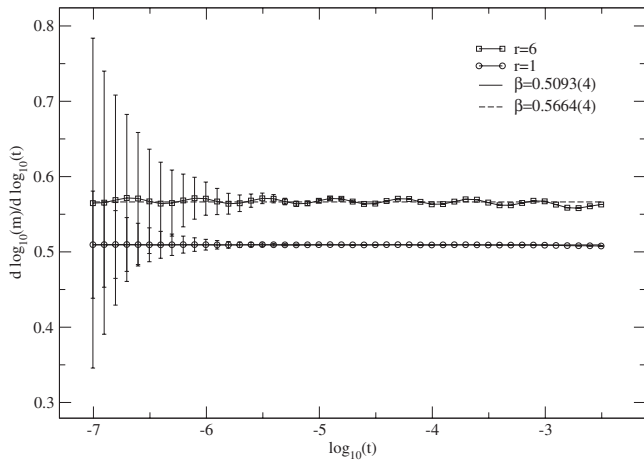


FIG. 4. Logarithmic derivative of the magnetization as a function of the reduced temperature for the period-doubling sequence, with $r=1$ and 6 . The log-periodic behavior is clearly identified, and the values for β are different from the classical value $1/2$. The continuous line and circles (traced line and squares) represent $r=1$ ($r=6$), and the length of the sequences is 2^{32} .

Fibonacci sequence (Fig. 3), it is clear that the aperiodic modulation does not change the exponent β from its classical value, $1/2$. Note the behavior of the log-derivative when the size of the Bethe lattice (or, equivalently, of the aperiodic sequence) is increased: for large values of the reduced temperature t , we are outside of the scaling region and Eq. (10) is not expected to be obeyed. When the reduced temperature is decreased, the correlation length increases and eventually it grows bigger than the number of generations of the Bethe lattice, leading to the finite-size effect seen in the left-hand side of the curves (they decrease from the classical value $1/2$). When the size of the lattice is increased, this departure from the infinite-size behavior takes place at lower values of t , as expected. This finite-size effect is also represented by the size of the error bars, as depicted in Figs. 4, 6, and 7 (we will comment more on this aspect below). The same overall behavior is obtained for γ and δ , which assume their classical values, 1 and 3 , respectively.

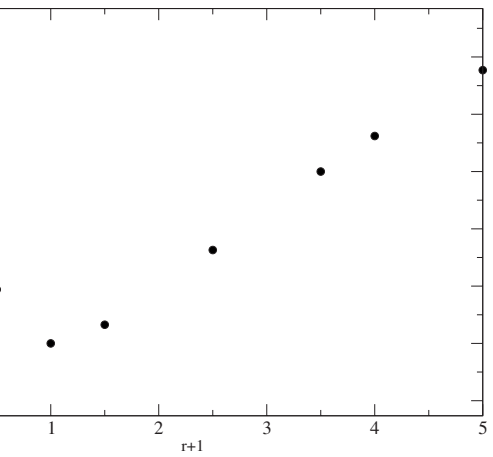
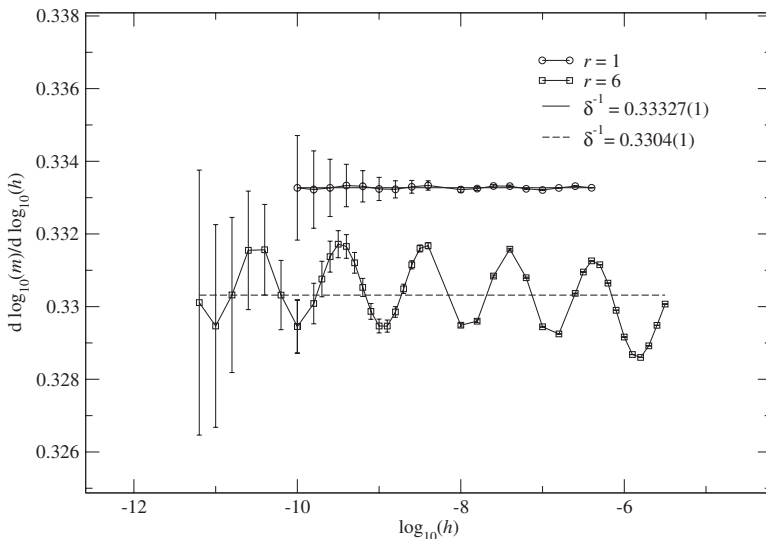


FIG. 5. Dependence of the exponent β on the parameter r . The homogeneous system is represented by $r=0$.

The scenario is more interesting for the period-doubling sequence. Although the sequences we need to generate are bigger than in the Fibonacci case, we were able to determine precise values for the exponents β , γ , and δ , which depend on the interaction ratio r . In Fig. 4 we show the logarithmic derivative of \bar{m} as a function of $\log_{10} t$ for $r=1$ and 6 . The scaling region spans three decades, and the exponent is clearly different from the classical value $1/2$: a fitting of the data to a log-periodic function leads to $\beta=0.5093(4)$ and $0.5664(5)$ for $r=1$ and 6 , respectively. The exponent β depends on the ratio r , as depicted in Fig. 5: the overall behavior is the same as that found in Ref. [20], but for a different sequence. There is a symmetry with respect to $r=0$: the exponent β is the same for $1+r=l$ and $1+r=1/l$, i.e., it is invariant with respect to the exchange $J_A \leftrightarrow J_B$. The exponent β increases for the aperiodic model, when compared to its uniform value. This is consistent with the following picture. The new (aperiodic) fixed points, in the renormalization-group sense, must have a greater value of ν than that for the uniform model (ν_0). Therefore, assuming $\nu=1/y_t$ to be valid, where y_t is the usual temperature scaling exponent in the renormalization-group equations, y_t is smaller for the aperi-

FIG. 6. Logarithmic derivative of the magnetization as a function of the magnetic field, at $T=T_c$, for the period-doubling sequence, with $r=1$ and 6 . The exponent $1/\delta$ is different from the classical value $1/3$ and depends on r . The continuous line and circles (traced line and squares) represent $r=1$ ($r=6$), and the length of the sequences is 2^{36} .

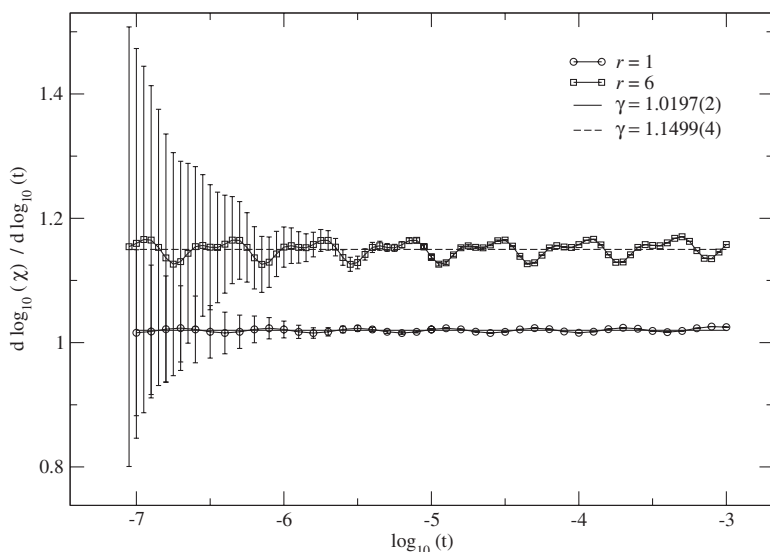


FIG. 7. Logarithmic derivative of the susceptibility as a function of the reduced temperature for the period-doubling sequence, with $r=1$ and 6 . The exponent γ is different from the classical value 1 and depends on r . The continuous line and circles (traced line and squares) represent $r=1$ ($r=6$), and the length of the sequences is 2^{32} .

odic model. But $\beta=(d-y_h)/y_t$ (y_h is the field scaling exponent) and y_h is seen to vary very little with r . In fact, our results for δ (see Fig. 6), given by $y_h/(d-y_h)$, show that, although this exponent depends on r for the period-doubling sequence, it varies much less than β . So the variation of this exponent is linked almost entirely to y_t and it increases for the aperiodic model (in fact, this picture is also observed for random disorder). In Fig. 7, we depict $d(\log_{10} \chi)/d(\log_{10} t)$ as a function of $\log_{10} t$, for $r=1$ and 6 : again, the dependence on r is evident. This figure makes clear the need of a second harmonic, to adjust the curve for $r=6$. The increase of the error bars for small values of $\log_{10}(t)$ in Figs. 4, 6, and 7 is a finite-size effect: the smaller t , the greater the correlation length, and the size of the Bethe lattice (or, equivalently, the size of the aperiodic sequence) must be increased, to reach the limit where the infinite-volume behavior is obtained.

We summarize our results in Table I, where we show the values of the three exponents for the period-doubling sequence and $r=0, 1$, and 6 . For $r=0$ we show the exact results on the Bethe lattice: our numerical values agree with them within error bars. The exponents clearly depend on r ; nevertheless, the usual relation among the three exponents, namely, $\beta=\gamma/(\delta-1)$, is satisfied for $r=1$ and is just satisfied for $r=6$ (see rows 3 and 4 in Table I).

TABLE I. Values for the exponents β , γ , and δ for the period-doubling sequence, for $r=0, 1$, and 6 . In the third and fourth rows we show evaluations of β , calculated from γ and δ and directly from our data, respectively. Note that the results in the first column are exact ones: our numerical values agree with them, within error bars. Numbers inside parentheses are errors in the last decimal figures.

	$r=0$	$r=1$	$r=6$
γ	1	1.0197(2)	1.1499(4)
δ	3	3.0006(2)	3.0266(9)
$\gamma/(\delta-1)$	1/2	0.5097(2)	0.5674(5)
β	1/2	0.5093(4)	0.5664(5)

Preliminary results obtained for the spin-1 Ising model lead to exactly the same behavior as for its spin-1/2 counterpart. As the former model is the Blume-Capel (BC) one with zero crystal field Δ and the latter corresponds to the BC model with $\Delta=-\infty$, we can infer that our results hold for the BC Hamiltonian and for all values of Δ such that the transition is continuous in the uniform case. Also, the results we obtain for the Blume-Capel model are not expected to depend on which interaction the disorder acts on, according to the following renormalization-group reasoning. Even if, initially, only the exchange constants follow an aperiodic sequence, the crystal field will not be uniform in the coarse-grained system, which is equivalent to the original one.

As a final note, we would like to mention that Eq. (2) assumes that the relation between the number of letters A and B at stage $n+1$ of the iteration process depends linearly on the number of letters A and B at stage n , namely,

$$N_A^{(n+1)} = aN_A^{(n)} + bN_B^{(n)}, \quad N_B^{(n+1)} = cN_A^{(n)} + dN_B^{(n)}, \quad (13)$$

where a, b, c , and d are the elements of the matrix \mathcal{M} . The former relation does not hold true for the Bethe lattice, since the letters A on different generations of the lattice do not lead to the same number of letters A and B (the same is true for the substitution of letters B) in the inflation process. Thus, as long as we cannot define the substitution matrix for aperiodic sequences on Bethe lattices, we are not able to determine the exponent ω in these cases.

In summary, we have studied the Ising model on a Bethe lattice, in the mean-field limit, such that the exchange constants follow two different aperiodic sequences: the Fibonacci and period-doubling ones. The results are as follows. For the Fibonacci sequence, the exponents are classical, namely, $\beta=1/2$, $\gamma=1$, and $\delta=3$. For the period-doubling sequence the exponents depend on the ratio of the interaction constants, but satisfy the usual relation among critical exponents for any value of r . Therefore, we show that mean-field-like procedures may lead to exponent values different from the classical ones. In this particular case, although thermal

fluctuations are suppressed by the mean-field character of the procedure we use, geometrical fluctuations present in the sequences are treated exactly. Preliminary results for lattices with finite coordination number z show the same qualitative behavior as for the mean-field limit ($z \rightarrow \infty$).

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